# Mathematics-II (25SMT-125)

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## Unit-2

**Chapter 1 : Vector Space** - Vector Space, Basis and dimension of a vector space.

**Chapter 2 : Linear Transformations** - Linear transformations (maps),range and kernel of a linear map, rank and nullity, Inverse of a linear transformation, rank- nullity theorem (Without Proof) , composition of linear maps, Matrix associated with a linear map.

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# Chapter 1

# Vector Space

**Aim:** To understand the concepts of Vector Space, Basis, and Dimension in a rigorous yet intuitive way.

## 1.1 Vector Space

Imagine you are at a playground. You can move forward/backward or left/right. These two directions are enough to reach any point on the ground. This playground is like a **vector space**.

**Definition**

A **Vector Space** $V$ over a field $F$ (for example $\mathbb{R}$ or $\mathbb{C}$) is a set together with two operations:
1. **Vector addition:** If $u, v \in V$, then $u + v \in V$. 2. **Scalar multiplication:** If $c \in F$ and $v \in V$, then $cv \in V$.
These operations must satisfy: - Commutativity: $u + v = v + u$ - Associativity: $(u + v) + w = u + (v + w)$ - Existence of zero vector: $0 + v = v$ - Existence of additive inverse: For $v$, there exists $-v$ such that $v + (-v) = 0$ - Distributivity of scalar multiplication over addition - Compatibility of scalar multiplication

**Examples**

1. The set $\mathbb{R}^2$ (all ordered pairs of real numbers) with usual addition and multiplication is a vector space. 2. The set of polynomials of degree $\leq n$ with real coefficients forms a vector space. 3. The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ forms a vector space.

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## 1.2 Basis of a Vector Space

Think of building with Lego blocks. You do not need every block in the store, just a minimal set of blocks that can create everything. These minimal building blocks in a vector space are called the **basis**.

**Definition**

A **basis** of a vector space $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that:
1. They are linearly independent. 2. They span $V$, i.e. every $v \in V$ can be written as a linear combination of them.

**Example**

In $\mathbb{R}^2$, the set $\{(1,0),(0,1)\}$ is a basis. For any $(x,y) \in \mathbb{R}^2$, we can write:

$$(x,y) = x(1,0) + y(0,1)$$

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## 1.3 Dimension of a Vector Space

The number of Lego blocks in your minimal set is like the number of **basis vectors**. This number is called the **dimension**.

**Definition**

The **dimension** of a vector space $V$ is the number of vectors in any basis of $V$.

**Examples**

1. $\mathbb{R}^2$ has dimension 2. 2. The space of polynomials of degree $\leq n$ has dimension $n+1$. 3. The space of all $m \times n$ matrices with real entries has dimension $mn$.

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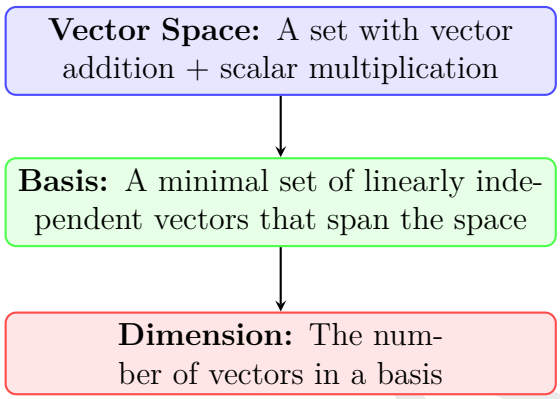
## 1.4 Visual Flowchart


Vector Space: A set with vector addition + scalar multiplication
Basis: A minimal set of linearly independent vectors that span the space
Dimension: The number of vectors in a basis

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## 1.5 Solved Example

**Example**

**Problem:** Find the dimension of the space of polynomials of degree $\leq 2$.
**Solution:** A general polynomial is $a_0 + a_1x + a_2x^2$. The set $\{1, x, x^2\}$ spans the space and is linearly independent. Thus, dim $= 3$.

## 1.6 Solved Problems on Vector Spaces, Basis and Dimension

### Problem 1

Verify whether the set $\{(1,2),(2,4)\}$ is linearly independent in $\mathbb{R}^2$.

**Solution:** Let $a(1,2)+b(2,4)=(0,0)$. This gives the equations:

$$a+2b=0, \quad 2a+4b=0.$$

From the first equation: $a=-2b$. Substituting into the second: $2(-2b)+4b=0 \Rightarrow 0=0$. Thus, there exists a non-trivial solution. Hence, the vectors are linearly **dependent**.

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### Problem 2

Show that $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$.

**Solution:** Any $(x,y,z)\in\mathbb{R}^3$ can be written as

$$(x,y,z)=x(1,0,0)+y(0,1,0)+z(0,0,1).$$

Thus, the given vectors span $\mathbb{R}^3$. Also, no vector is a linear combination of the others, so they are linearly independent. Therefore, this set forms a **basis**.

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### Problem 3

Find the dimension of the space of all polynomials of degree $\leq 4$.

**Solution:** A general polynomial is

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4.$$

The basis is $\{1,x,x^2,x^3,x^4\}$. Therefore, the dimension is 5.

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### Problem 4

Find the dimension of the space of $2\times 3$ real matrices.

**Solution:** Each $2\times 3$ matrix has 6 entries, each of which can take independent real values. Hence, the dimension of the space is $2\cdot 3=6$.

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## Problem 5

Decide if $\{1+x,\, x+x^2,\, 1+x^2\}$ forms a basis of the space of polynomials of degree $\leq 2$.

**Solution:** Consider a linear combination:

$$a(1+x) + b(x+x^2) + c(1+x^2) = 0.$$

Expanding:

$$(a+c) + (a+b)x + (b+c)x^2 = 0.$$

Equating coefficients:

$$a+c=0, \quad a+b=0, \quad b+c=0.$$

The only solution is $a = b = c = 0$. Hence, the vectors are linearly independent. Since the dimension of the space is 3 and we have 3 independent vectors, this is a basis.

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## Problem 6

Find a basis of the solution space of $x_1 + x_2 + x_3 = 0$.

**Solution:** Let $x_1 = -x_2 - x_3$. General solution: $(-x_2 - x_3,\, x_2,\, x_3)$. This can be written as

$$x_2(-1,1,0) + x_3(-1,0,1).$$

Hence, a basis is $\{(-1,1,0), (-1,0,1)\}$ and the dimension is 2.

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## Problem 7

Prove that $\{(1,1,0), (1,0,1), (0,1,1)\}$ is linearly independent.

**Solution:** Consider

$$a(1,1,0) + b(1,0,1) + c(0,1,1) = (0,0,0).$$

This gives equations:

$$a+b=0, \quad a+c=0, \quad b+c=0.$$

From $b + c = 0 \Rightarrow b = -c$. Substitute in $a + b = 0 \Rightarrow a = -b$. Then $a + c = 0 \Rightarrow -b + c = 0 \Rightarrow c = b$. Thus $b = c = 0$, hence $a = 0$. Therefore, the vectors are linearly independent.

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## Problem 8

Find the dimension of the space of symmetric $2 \times 2$ matrices.

**Solution:** A general symmetric matrix is

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

There are three free parameters $(a, b, c)$. Hence, the dimension is 3. A basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

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## Problem 9

Find the dimension of the space of skew-symmetric $3 \times 3$ matrices.

**Solution:** A skew-symmetric matrix has the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

The free parameters are $a, b, c$. Thus, the dimension is 3.

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## Problem 10

Find the dimension of the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

**Solution:** The rank of $A$ is 2 (rows are independent). By the rank-nullity theorem:

$$\text{nullity} = n - \text{rank} = 3 - 2 = 1.$$

Hence, the null space has dimension 1.

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## Problem 11

Show that $\{(1,0,-1),(0,1,1),(1,1,0)\}$ forms a basis of $\mathbb{R}^3$.

**Solution:** Consider

$$a(1,0,-1)+b(0,1,1)+c(1,1,0)=(0,0,0).$$

This gives equations:

$$a+c=0, \quad b+c=0, \quad -a+b=0.$$

From $-a+b=0 \Rightarrow b=a$. From $a+c=0 \Rightarrow c=-a$. Substituting: $b+c=a-a=0$. Thus $a=b=c=0$. Therefore, the set is linearly independent. Since there are 3 vectors in $\mathbb{R}^3$, they form a basis.

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## Problem 12

Find the dimension of the subspace of $\mathbb{R}^4$ spanned by $\{(1,1,0,0),(0,1,1,0),(0,0,1,1),(1,0,0,1)\}$.

**Solution:** Arrange vectors as rows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Row-reducing, we find rank = 3. Hence, the dimension of the span is 3. Therefore, these 4 vectors span a 3-dimensional subspace of $\mathbb{R}^4$.

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# Chapter 2

# Linear Transformations

**Aim:** To understand the concepts of Linear Transformations (maps), range, kernel, rank and nullity, and to develop skills in solving related problems.

## 2.1 Introduction

In many real-life situations, we want to study functions that preserve structure. For example: - Stretching or rotating a rubber sheet keeps straight lines straight. - Scaling an image enlarges or shrinks it but does not distort proportionality.

Such structure-preserving functions in vector spaces are called **linear transformations**.

## 2.2 Definition

A function $T : V \to W$ between vector spaces over the same field $F$ is called a **linear transformation** if for all $u, v \in V$ and scalars $c \in F$:

$$T(u + v) = T(u) + T(v), \quad T(cu) = cT(u).$$

**Key idea:** A linear transformation respects both vector addition and scalar multiplication.

## 2.3 Examples

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$.
2. Differentiation operator $D : P_n \to P_{n-1}$ where $D(f(x)) = f'(x)$.

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3. $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = 2x - y + 3z$.

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## 2.4 Kernel and Range

**Kernel (Null Space):**

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

It represents all vectors that are sent to the zero vector.

**Range (Image):**

$$\text{Range}(T) = \{T(v) \mid v \in V\} \subseteq W.$$

It is the set of all possible outputs.

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## 2.5 Rank and Nullity

**Rank:** Dimension of the range of $T$.

$$\text{rank}(T) = \dim(\text{Range}(T))$$

**Nullity:** Dimension of the kernel of $T$.

$$\text{nullity}(T) = \dim(\ker(T))$$

**Rank–Nullity Theorem:** For a linear map $T : V \to W$,

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

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## 2.6 Solved Problems

**Problem 1:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + y, x - y)$. Find $\ker(T)$ and $\text{Range}(T)$.

**Solution:** Let $(x, y) \in \mathbb{R}^2$. Then

$$T(x, y) = (x + y, x - y).$$

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For kernel: Solve $T(x, y) = (0, 0)$. That gives equations:

$$x + y = 0, \quad x - y = 0.$$

Adding: $2x = 0 \implies x = 0$. Then $y = 0$. So $\ker(T) = \{(0, 0)\}$.

For range: Output is $(x + y, x - y)$. Let $u = x + y$, $v = x - y$. For any $(u, v) \in \mathbb{R}^2$, we can solve for $x, y$. So Range$(T) = \mathbb{R}^2$.

Hence rank$(T) = 2$, nullity$(T) = 0$.

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**Problem 2:** Check whether $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, 0, z)$ is linear.

**Solution:** Take $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$.

$$\begin{aligned} T((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2). \\ &= (x_1 + x_2, 0, z_1 + z_2). \\ &= (x_1, 0, z_1) + (x_2, 0, z_2). \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

Also scalar property holds:

$$T(c(x, y, z)) = (cx, 0, cz) = c(x, 0, z) = cT(x, y, z).$$

Thus $T$ is linear.

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**Problem 3:** Find $\ker(T)$ and rank of $T(x, y, z) = (x + z, y + z, z)$.

**Solution:** We write in matrix form:

$$T(x, y, z) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Call this matrix $A$. Rank of $T$ = rank of $A$.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Row-reduction: Already upper triangular with 3 pivots. So rank$(T) = 3$.

For kernel: Solve $A[x, y, z]^T = 0$. That gives $x + z = 0,\ y + z = 0,\ z = 0$. So $x = 0, y = 0, z = 0$. Thus $\ker(T) = \{(0, 0, 0)\}$. Nullity=0.

Hence $\dim(V) = 3 = \text{rank} + \text{nullity} = 3 + 0$.

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**Problem 4:** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, y, x + y)$. Find basis of Range$(T)$ and $\ker(T)$.

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**Solution:** Matrix form:

$$T(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

So columns are $(1,0,1)^T$ and $(0,1,1)^T$. Range is span of these two vectors. Check independence: $a(1,0,1) + b(0,1,1) = (0,0,0)$ gives $a = 0, b = 0$. So they are independent.

Therefore Range$(T) = \text{span}\{(1,0,1),(0,1,1)\}$. Basis of range given. Rank=2.

For kernel: Solve $T(x,y) = (0,0,0)$. That means $(x, y, x+y) = (0,0,0)$. So $x = 0, y = 0$. Kernel $= \{(0,0)\}$. Nullity=0.

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**Problem 5:** Let $T : P_2 \to P_2$ be differentiation: $T(f(x)) = f'(x)$. Find rank and nullity.

**Solution:** Any $f(x) = a_0 + a_1 x + a_2 x^2$.

$$T(f(x)) = a_1 + 2a_2 x.$$

Range: all polynomials of degree $\leq 1$. So rank$(T) = 2$.

Kernel: Solve $f'(x) = 0 \implies a_1 = 0, a_2 = 0$. So $f(x) = a_0$. Kernel = constant polynomials, which form 1-dimensional space. Nullity=1.

Check theorem: $\dim(P_2) = 3 = 2 + 1$.

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**Problem 6:** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z) = (x+y, y+z)$. Find rank and nullity.

**Solution:** Matrix:

$$T(x,y,z) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Rank: Row-reduction $\to$ independent rows. So rank=2.

Nullity $= \dim(V) - \text{rank} = 3 - 2 = 1$.

Kernel: Solve $x + y = 0,\ y + z = 0$. So $x = -y,\ z = -y$. Kernel $= \text{span}\{(-1,1,-1)\}$.

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## 2.7 Recap

- A **linear transformation** preserves vector addition and scalar multiplication. - The **kernel** is the set of inputs mapped to zero. - The **range** is the set of outputs. - **Rank** = dimension of range. - **Nullity** = dimension of kernel. - Rank–Nullity theorem: $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.

## 2.8 Inverse of a Linear Transformation

**Definition:** A linear transformation $T : V \to W$ is said to be **invertible** if there exists another linear transformation $S : W \to V$ such that

$$S(T(v)) = v \quad \forall v \in V,$$
$$T(S(w)) = w \quad \forall w \in W.$$

In that case, $S$ is called the **inverse** of $T$ and denoted by $T^{-1}$.

**Condition for existence:** - $T$ is invertible if and only if it is **bijective** (one–one and onto). - For finite-dimensional spaces, $T$ is invertible if and only if $\text{rank}(T) = \dim(V) = \dim(W)$.

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**Problem 7:** Check if $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x + y, x + 3y)$ is invertible.

**Solution:** Matrix representation:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Determinant:

$$\det(A) = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5 \neq 0.$$

So $A$ is invertible. Thus $T$ is invertible.

Inverse matrix:

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Hence $T^{-1}(u, v) = \left(\frac{3u-v}{5}, \frac{-u+2v}{5}\right)$.

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**Problem 8:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x + y, x + y)$. Is $T$ invertible?

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**Solution:** Matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$\det(A) = 1 \cdot 1 - 1 \cdot 1 = 0$. So matrix is not invertible. Therefore $T$ is not invertible.

In fact, Range$(T) = \text{span}\{(1, 1)\}$, so rank$= 1 < 2$.

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**Problem 9:** Check whether differentiation operator $D : P_1 \to P_1$ defined by $D(f(x)) = f'(x)$ is invertible.

**Solution:** Basis of $P_1$: $\{1, x\}$. For $f(x) = a + bx$, $D(f(x)) = b$.

So range $= \text{span}\{1\}$ (only constants). $\dim(P_1) = 2$, but rank$= 1$. So not invertible.

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## 2.9 Rank–Nullity Theorem

**Statement (without proof):** If $T : V \to W$ is a linear transformation from a finite-dimensional vector space $V$, then

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

This result connects the input space dimension with the dimensions of the range and kernel.

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**Problem 10:** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x + y, y + z)$. Verify Rank–Nullity theorem.

**Solution:** Matrix form:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Rank: Rows independent $\implies$ rank=2. $\dim(\mathbb{R}^3) = 3$. So nullity $= 3 - 2 = 1$.

Kernel: Solve $(x + y, y + z) = (0, 0)$. That gives $x = -y, z = -y$. Kernel $= \text{span}\{(-1, 1, -1)\}$, dimension=1.

Hence rank+nullity $= 2 + 1 = 3 = \dim(V)$. Verified.

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**Problem 11:** Let $T : P_2 \to P_2$ be $T(f(x)) = f'(x)$. Verify Rank–Nullity theorem.

**Solution:** $\dim(P_2) = 3$.

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Kernel: $f'(x) = 0 \implies f(x) = a$ (constant). So $\ker(T)$ has dimension 1. Nullity=1.

Range: $f'(x) = a_1 + 2a_2x$, i.e. degree $\leq 1$. So rank=2.

Rank + Nullity $= 2 + 1 = 3 = \dim(P_2)$. Verified.

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**Problem 12:** Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, y, 0)$. Verify Rank–Nullity theorem.

**Solution:** Matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Columns are independent $\implies$ rank=2.

$\dim(\mathbb{R}^2) = 2$. So nullity $= 2 - 2 = 0$.

Kernel $= \{(0, 0)\}$, indeed dimension 0.

Thus rank+nullity $= 2 + 0 = 2 = \dim(V)$. Verified.

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## 2.10 Recap

- A linear map is **invertible** if it is bijective. - For finite-dimensional spaces, invertibility $\iff$ determinant $\neq 0$ (matrix form). - The **Rank–Nullity Theorem** relates input dimension, rank, and nullity:

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

- Every problem must check consistency by verifying this identity.

# Composition of Linear Maps

**Definition:** Let $U, V$, and $W$ be vector spaces over the same field $\mathbb{F}$. Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are linear maps. Then the composition $T_2 \circ T_1 : U \to W$ is defined as

$$(T_2 \circ T_1)(u) = T_2(T_1(u)) \quad \forall u \in U$$

**Properties:**

- Composition of linear maps is again a linear map.
- If $T_1$ and $T_2$ are both one-one (injective), then $T_2 \circ T_1$ is also one-one.

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- If $T_1$ and $T_2$ are both onto (surjective), then $T_2 \circ T_1$ is also onto.

**Real-life analogy:** Think of $T_1$ as a machine that converts wheat into flour, and $T_2$ as another machine that converts flour into bread. The composition $T_2 \circ T_1$ is like a single machine that directly converts wheat into bread.

### Solved Examples

**Example 1:** Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_1(x, y) = (x + 1, y)$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_2(x, y) = (2x, 3y)$. Find $(T_2 \circ T_1)(x, y)$.

**Solution:**

$$T_1(x, y) = (x + 1, y)$$

$$T_2(T_1(x, y)) = T_2(x + 1, y) = (2(x + 1), 3y) = (2x + 2, 3y)$$

Hence,

$$(T_2 \circ T_1)(x, y) = (2x + 2, 3y)$$

**Example 2:** Check whether $T_2 \circ T_1 = T_1 \circ T_2$ for the above maps.

**Solution:** We already have $(T_2 \circ T_1)(x, y) = (2x + 2, 3y)$.

Now,

$$T_2(x, y) = (2x, 3y), \quad T_1(2x, 3y) = (2x + 1, 3y)$$

Thus,

$$(T_1 \circ T_2)(x, y) = (2x + 1, 3y)$$

Clearly,

$$(T_2 \circ T_1)(x, y) \neq (T_1 \circ T_2)(x, y)$$

So, composition is not commutative in general.

## Matrix Associated with a Linear Map

**Definition:** Every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix $A$ such that:

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

**Procedure to find the matrix of $T$:**

1. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{R}^n$.
2. Compute $T(e_i)$ for each basis vector.
3. Express each $T(e_i)$ as a column in the matrix.
4. The resulting $m \times n$ matrix is the matrix representation of $T$.

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## Solved Examples

**Example 3:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (x + 2y, 3x + y)$$

Find the matrix representation of $T$ with respect to the standard basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$.

**Solution:** Step 1: Compute $T(e_1)$ and $T(e_2)$.

$$T(1, 0) = (1 + 0, 3 \cdot 1 + 0) = (1, 3)$$
$$T(0, 1) = (0 + 2, 0 + 1) = (2, 1)$$

Step 2: Arrange them as columns.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Thus, the matrix of $T$ is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

**Example 4:** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = (x + 2y + z,\ 3x - y)$$

Find the matrix associated with $T$.

**Solution:** Step 1: Apply $T$ to basis vectors.

$$T(1, 0, 0) = (1, 3), \quad T(0, 1, 0) = (2, -1), \quad T(0, 0, 1) = (1, 0)$$

Step 2: Arrange as columns in a $2 \times 3$ matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

Hence, $A$ is the matrix representation of $T$.

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# Solved Problems

**Problem 1:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + 2y, 3x - y)$. Check whether $T$ is linear.

**Solution:** Check addition:

$$T((x_1, y_1)+(x_2, y_2)) = T(x_1+x_2, y_1+y_2) = ((x_1+x_2)+2(y_1+y_2), 3(x_1+x_2)-(y_1+y_2))$$

$$= (x_1 + 2y_1, 3x_1 - y_1) + (x_2 + 2y_2, 3x_2 - y_2) = T(x_1, y_1) + T(x_2, y_2)$$

Check scalar multiplication:

$$T(c(x, y)) = T(cx, cy) = (cx + 2cy, 3cx - cy) = c(x + 2y, 3x - y) = cT(x, y)$$

Hence $T$ is linear.

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**Problem 2:** Find $\ker(T)$ and Range$(T)$ for $T(x, y) = (x + 2y, 3x - y)$.

**Solution:** Kernel: Solve $T(x, y) = (0, 0)$

$$x + 2y = 0, \quad 3x - y = 0$$

Solve simultaneously: $y = 3x$, substitute: $x + 2(3x) = 7x = 0 \implies x = 0, y = 0$

$$\ker(T) = \{(0, 0)\}, \text{nullity} = 0$$

Range: Matrix form

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

Determinant $\det(A) = 1\cdot(-1)-2\cdot 3 = -1-6 = -7 \neq 0$ Rank= 2, Range= $\mathbb{R}^2$

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**Problem 3:** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x + y, y + z, x + z)$. Find rank and nullity.

**Solution:** Matrix form:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Row-reduce or check linear independence: 3 rows independent $\implies$ rank=3 $\dim(\mathbb{R}^3) = 3$, nullity= $3 - 3 = 0$

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**Problem 4:** Check if $T : P_2 \to P_2$, $T(f(x)) = f'(x)$ is invertible.

**Solution:** $P_2$ basis: $\{1, x, x^2\}$ $D(1) = 0, D(x) = 1, D(x^2) = 2x$

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Kernel: $f'(x) = 0 \implies f(x) = a$, dim=1 Range: all degree $\leq 1$, dim=2
Since $\dim(P_2) = 3 \neq \text{rank} = 2$, $T$ is not invertible.

—

**Problem 5:** Find the inverse of $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + y, x - y)$.

**Solution:** Matrix: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Determinant $\det(A) = -1-1 = -2 \neq 0$ $\to$ invertible

Inverse:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence $T^{-1}(u, v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$

—

**Problem 6:** Verify rank-nullity theorem for $T(x, y, z) = (x + z, y + z)$

**Solution:** Matrix: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Rank=2, $\dim(\mathbb{R}^3) = 3$ Nullity $= 3 - 2 = 1$ Kernel: Solve $x + z = 0, y + z = 0 \implies x = -z, y = -z \to$ dimension 1 Rank+Nullity=2+1=3.

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**Problem 7:** Composition: $T_1(x, y) = (x + y, x)$, $T_2(x, y) = (2x, y - x)$. Compute $T_2 \circ T_1$.

**Solution:**

$$T_2(T_1(x, y)) = T_2(x + y, x) = (2(x + y), x - (x + y)) = (2x + 2y, -y)$$

—

**Problem 8:** Check if $T_2 \circ T_1 = T_1 \circ T_2$ for above maps.

**Solution:**

$$T_1(T_2(x, y)) = T_1(2x, y - x) = (2x + (y - x), 2x) = (x + y, 2x)$$

Clearly $T_1 \circ T_2 \neq T_2 \circ T_1$

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**Problem 9:** Matrix associated with $T(x, y) = (3x - 2y, x + y)$

**Solution:** Basis $e_1 = (1, 0), e_2 = (0, 1)$ $T(e_1) = (3, 1), T(e_2) = (-2, 1)$ Matrix:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

—

**Problem 10:** Matrix associated with $T(x, y, z) = (x + z, y - z)$

**Solution:** $e_1 = (1, 0, 0) \to T(e_1) = (1, 0)$ $e_2 = (0, 1, 0) \to T(e_2) = (0, 1)$ $e_3 = (0, 0, 1) \to T(e_3) = (1, -1)$

Notefy

Matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

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**Problem 11:** Find $\ker(T)$ if $T(x, y, z) = (x + y + z, x - y + 2z)$
**Solution:** Solve:

$$x + y + z = 0, \quad x - y + 2z = 0$$

Add equations: $2x + 3z = 0 \implies x = -3z/2$ From first: $-3z/2 + y + z = 0 \implies y = z/2$ Kernel: span$\{(-3/2, 1/2, 1)\}$, dim=1 $\rightarrow$ nullity=1

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**Problem 12:** Verify rank-nullity theorem for $T(x, y, z) = (x - y, y - z)$
**Solution:** Matrix: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$
Rank: Rows independent $\rightarrow$ rank=2 $\dim(V) = 3 \rightarrow$ nullity $= 3 - 2 = 1$
Kernel: Solve $x - y = 0, y - z = 0 \implies x = y = z \rightarrow$ span$\{(1, 1, 1)\}$ Rank+Nullity=2+1=3

Notefy

# MST 2 - Practice Sheet

**General Instructions:** Attempt all questions
**Total Time:** 1 hour
**Total Marks:** 20

| Q.No | Statement |
|---|---|
| | **Section A** |
| 1 | Define a vector space over a field. Give two examples of vector spaces other than $\mathbb{R}^n$. |
| 2 | Explain the concepts of basis and dimension of a vector space with examples. |
| 3 | Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + 2y, 3x - y)$. Determine $\ker(T)$ and Range$(T)$. |
| 4 | Let $T : P_2 \to P_2$ be defined by $T(f(x)) = f'(x)$. Find the kernel and rank of $T$. |
| 5 | State the Rank-Nullity theorem and explain it with a simple example. |
| | **Section B** |
| 6 | Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + y, x - y)$. Determine if $T$ is invertible. If yes, find $T^{-1}$. |
| 7 | Let $T_1(x, y) = (x + y, x)$ and $T_2(x, y) = (2x, y - x)$. Compute $(T_2 \circ T_1)(x, y)$ and $(T_1 \circ T_2)(x, y)$. Are they equal? |

Noneify

# Answers

### Section A

1. **Vector Space:**
   A **vector space** $V$ over a field $\mathbb{F}$ is a set of objects (vectors) with operations of vector addition and scalar multiplication satisfying: closure, associativity, commutativity, existence of zero vector, existence of additive inverse, distributivity, and compatibility of scalars.

   Examples: 1. The set of all $2 \times 2$ matrices over $\mathbb{R}$. 2. The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$.

2. **Basis and Dimension:**
   **Basis:** A set of vectors in $V$ that is linearly independent and spans $V$.
   **Dimension:** Number of vectors in any basis.
   Example: $\mathbb{R}^2$, basis $= \{(1,0),(0,1)\}$, dimension $= 2$.

3. **Kernel and Range:** $T(x,y) = (x+2y, 3x-y)$
   Kernel: Solve $T(x,y) = (0,0)$

   $$x + 2y = 0, \quad 3x - y = 0$$

   Solve: $y = 3x \implies x + 2(3x) = 7x = 0 \implies x = 0, y = 0$

   $$\ker(T) = \{(0,0)\}, \text{nullity} = 0$$

   Range: Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\det(A) = -7 \neq 0$ Rank=2 $\to$ Range= $\mathbb{R}^2$

4. **Derivative Map $T(f) = f'$:** Basis of $P_2 = \{1, x, x^2\}$
   Kernel: $f'(x) = 0 \implies f(x) = c$, dim=1 Range: Span$\{1, x\}$, dim=2 $\to$ rank=2

5. **Rank-Nullity Theorem:** For $T : V \to W$, $\dim(V) = \text{rank}(T) + \text{nullity}(T)$
   Example: $T(x,y,z) = (x+y, y+z)$, rank=2, nullity=1, $\dim(\mathbb{R}^3) = 3$, satisfies theorem.

### Section B

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6. $T(x, y) = (x + y, x)$

   Matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\det(A) = -2 \neq 0 \implies T$ invertible

   $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

   $$T^{-1}(u, v) = \left(\frac{u + v}{2}, \frac{u - v}{2}\right)$$

7. Composition: $T_1(x, y) = (x + y, x)$, $T_2(x, y) = (2x, y - x)$

   $T_2 \circ T_1(x, y) = T_2(x + y, x) = (2(x + y), x - (x + y)) = (2x + 2y, -y)$

   $T_1 \circ T_2(x, y) = T_1(2x, y - x) = (2x + y - x, 2x) = (x + y, 2x)$

   Not equal $\rightarrow$ composition not commutative.

*For more resources, visit https://subhayu18.github.io/Noteify/*

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